CS201

Mathematics For Computer Science

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Group Number: 5

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Assignment 2

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Question 1

A **partition** of n objects is a collection of its mutually disjoint subsets, called blocks, whose union gives the whole set. Let $S(n; k_1, k_2, \ldots, k_n)$ denote the number of all partitions of n objects with k_i i-element blocks (i.e., $k_1 + 2k_2 + \cdots + nk_n = n$). In other words,

$$k_i = \text{the number of } i\text{-element blocks in a partition}$$

Show that $S(n; k_1, k_2, \ldots, k_n) = \dfrac{n!}{k_1!k_2!\ldots k_n!(1!)^{k_1}(2!)^{k_2}\ldots(n!)^{k_n}}$.

Solution

Let $\mathbb{S}$ denote the set of n objects given in question. Since, k_i i-element blocks are different for different $i \in [1, n]$, we can construct distinct partition with k_i i-element blocks by first choosing k_1 1-element blocks constructed from n objects of $\mathbb{S}$, then k_2 2-element block constructed from left $n - 1.k_1$ objects of $\mathbb{S}$, and lastly k_n n-element block constructed from left objects. Following this construction we can calculate the number of partitions of given type, with the help of following observation:

Observation 1.1. Number of ways to choose a set of k i-element blocks from set $\mathbb{A} \subseteq \mathbb{S}$ s.t. $ki \leq a$ and all i-element blocks are mutually exclusive and elements of each i-element block are distinct, is given by

$$\frac{a!}{(k!)(i!)^k(a-ki)!}, \text{ where } a = |\mathbb{A}|$$

Proof. The number of ways of choosing ordered tuple of k i-element blocks from $\mathbb{A}$

= (Number of choosing i elements for 1^{st} i-element block from a elements)(Number of choosing i elements for 2^{nd} i-element block from left $a-i$ elements)...(Number of choosing i elements for k^{th} i-element block from $a-(k-1)i$ elements)

$$
\begin{aligned}
&= \binom{a}{i}\binom{a-i}{i}\dots\binom{a-(k-1)i}{i} \\
&= \frac{a!}{i!(a-i)!}\frac{(a-i)!}{i!(a-2i)!}\frac{(a-(k-1)i)!}{i!(a-ki)!} \\
&= \frac{a!}{(i!)^k(a-ik)!}
\end{aligned}
$$

Since, set of elements of ordered tuple will remain identical for all permutations of sets of tuple,

$$
\begin{aligned}
\text{number of sets} &= \frac{\text{number of tuples}}{\text{number of permutations}} \\
&= \frac{\frac{a!}{(i!)^k(a-ik)!}}{k!} \\
&= \frac{a!}{(k!)(i!)^k(a-ik)!}
\end{aligned}
$$

□

$S(n;k_1,k_2,\dots,k_n)$ = (No. of ways of choosing k_1 1-element block from n elements)(No. of ways of choosing k_2 2-element block from $n-k_1$ elements)...(No. of ways of choosing k_n n-element block from $n-\sum_{j=1}^{n-1}k_j$)

$$
= (\frac{n!}{(k_1!)(1)^{k_1}(n-1.k_1)!})(\frac{(n-1.k_1)!}{(k_2!)(2!)^{k_2}(n-1.k_1-2.k_2)!})\dots(\frac{(n-\sum_{j=1}^{n-1}k_j\sum)!}{(k_n!)(n!)^{k_n}(n-\sum_{j=1}^{n}k_j)!})
$$

(Using Observation 1.1)

$$
= \frac{n!}{k_1!k_2!\dots k_n!(1!)^{k_1}(2!)^{k_2}\dots(n!)^{k_n}}
$$

Question 2

Show that for every k, the product of any k consecutive natural numbers is divisible by $k!$.

Solution

It's trivial to see that the statement is true for $k = 0$, as product of zero consecutive natural numbers is zero and hence it is divisible by $k! = 1$. Now, we will prove the statement for $k \in \mathbb{N}$. Consider the number of possible ways of choosing k obejcts from $n + k$ objects where $n \in \mathbb{W}$ and $k \in \mathbb{N}$. We have the fact that $n + k \geq k$ for all $n \in \mathbb{W}$ and $k \in \mathbb{W}$ and hence the number of ways of choosing k objects from $n + k$ objects would be a natural number. Let's define these many number of ways as a function $f : \mathbb{W} \times \mathbb{N} \to \mathbb{N}$ such that

$$f(n, k) = \binom{n+k}{k} \tag{2.1}$$

Let us also define another function $p : \mathbb{W} \times \mathbb{N} \to \mathbb{N}$ which denotes the product of k consecutive natural numbers after a non-negative integer n as

$$p(n, k) = \prod_{i=1}^{k} (n + i) \tag{2.2}$$

Now, notice the fact that we can also write $f(n, k)$ equivalently as,

$$\begin{aligned}
f(n, k) &= \frac{(n+k)!}{n!k!} \\
&= \left(\frac{(n+k)!}{n!}\right) \frac{1}{k!} \\
&= \left(\prod_{i=1}^{k} (n+i)\right) \frac{1}{k!} \\
&= \frac{p(n, k)}{k!} \\
\Rightarrow p(n, k) &= k! * f(n, k)
\end{aligned}$$

The last equation and the fact that $f(n,k) \in \mathbb{N}$ implies that $k!$ divides $p(n,k)$ for all $n \in \mathbb{W}$ and $k \in \mathbb{N}$. Hence Proved. $\square$

Question 3

Show that the number of pairs (A, B) of distinct subsets of $\{1, 2, \ldots, n\}$ with $A \subset B$ is $3^n - 2^n$.

Solution

Let S be the set containing the first n natural numbers or

$$S = \{1, 2, \ldots, n\} \tag{3.1}$$

Consider the process of forming two subsets A and B out of set S.

Observation 1: Number of pairs (A, B) of distinct subsets of S with $A \subset B$ is equal to the number of pairs (A, B) with $A \subseteq B$ subtracted with number of pairs (A, B) with $A = B$.

Proof. This observation is direct result of the fact that the set of pairs (A, B) with $A \subseteq B$ equals the union of the set of pairs (A, B) with $A \subset B$ and the set of pairs (A, B) with $A = B$. If we consider the three sets discussed above as S_1, S_2 and S_3. We have

$$S_1 = S_2 \cup S_3 \tag{3.2}$$

$$S_2 \cap S_3 = \emptyset \tag{3.3}$$

Using (3.2) and (3.3), we get

$$n(S_1) = n(S_2) + n(S_3) - n(S_2 \cap S_3)$$
$$n(S_1) = n(S_2) + n(S_3)$$

where n(S) represents the cardinality of the set S. Therefore, the above equation implies that

$$n(S_2) = n(S_1) - n(S_3) \tag{3.4}$$

This is exactly what is the statement of Observation 1. Hence Proved. $\square$

Now, to proceed further in the proof we need to find the number of pairs (A, B) with

$A \subseteq B$ and the set of pairs (A, B) with $A = B$, which are easier to find. We will present a construction strategy for finding the number of such pairs. We will try constructing pairs of subsets with these properties and find how many we can construct.
Notice, the fact that each element of S has the following four options available for it.

1. Either it is goes in A and not in B.
2. Either it is goes in B and not in A.
3. It goes in both in A and B.
4. It goes neither in A nor in B.

Consider an arbitrary element of S, say x. Now, for the condition $A \subseteq B$ to be true, all options except (1) can be true, hence this element has 3 options available for it. And since x was any arbitrary element so every element in set S has 3 options available for it. Hence the number of pairs (A, B) with $A \subseteq B$ are equal to 3^n.
Similarly, for the condition $A = B$ to be true, the element x has all options available except (1) and (2), hence for this case this element has 2 options available for it. And since x was again any arbitrary element, so every element in set S has 2 options available for it. Hence, the number of such pairs are 2^n.

Now, using **Observation 1** we can say that the required the number of pairs is the subtraction of the number of pairs found above or $3^n - 2^n$. Hence Proved. $\square$

Question 4

There is a set of $2n$ people (n males and n females). A good party is a set with the same number of males and females. How many ways are there to build such a good party?

Solution

In each valid choosing of a good party out of $2n$ people, let us assume there are i men and i women. Since the number of men and women both vary for 0 to n, the value of i also varies between 0 and n.

Then, a good party is described by i men and i women in the set ($0 \leq i \leq n$) and $n - i$ men and $n - i$ women not in the set but belonging to the universal set of all people. Now, choosing the i men/women also leaves us with only 1 way to choose the remaining $n - i$ men/women who are not in the good party.

So, to define a good party, we choose i men to include in the set and $n - i$ women to not include in the set ($0 \leq i \leq n$). So, we choose $i + (n - i) = n$ people out of which i are men whom we place in our good party and $n - i$ women whom we reject / we choose the i women among the rest n people whom we haven't chosen explicitly. This means, sum of all the above possible cases is the answer that we seek.

In other words, we need to count the number of ways we can choose $i + (n - i) = n$ people from $2n$ people, which is precisely $\binom{2n}{n}$.

Question 5

1. Show that the number of integer solution to the equation

$$x_1 + x_2 + \cdots + x_n = k$$

under the condition that $x_i \geq 0$ for all i is $\binom{n+k-1}{k}$.

2. Let n and $k \geq l$ be positive integers. How many different integer solutions are there to the equation $x_1 + x_2 + \cdots + x_n = k$ such that $0 \leq x_i < l$ for all i.

Solution

1 5.1

Consider k identical boxes and $n-1$ identical bars. Since, we need n integers, we try to create n partitions by choosing $n-1$ positions for $n-1$ bars.
Each time we choose $n-1$ positions for $n-1$ bars, we create n partitions which in total have k boxes.
This arrangement can be described as perfect bijection to our original problem by defining the number of boxes in the i^{th} partition to be the non-negative integer x_i , $0 \leq x_i \leq k$.
Since when two partitions are adjacent, the number of boxes is 0 which is the minimum possible, and when there are k boxes in a a partition while others are empty which is the maximum value possible for the i^{th} integer x_i, this also satisfies the constraint imposed on the values of x_i.
So, we there is in total $k+n-1$ positions where we may place boxes or bars on each position. Each such choice is a unique configuration. Hence, the number of ways to choose $n-1$ positions for $n-1$ bars is $\binom{n+k-1}{n-1} = \binom{n+k-1}{k}$.

2 5.2

As defined in the lectures, Principle of Inclusion and Exclusion:

$$\bigcup_{i=1}^{n} A_i = \sum_{i=1}^{n} |A_i| - \sum_{1 \le i < j \le n} |A_i \cap A_j| + \sum_{1 \le i < j < k \le n} |A_i \cap A_j \cap A_k| - \ \ldots \ + (-1)^{n-1} \cdot |A_1 \cap A_2 \ldots A_n|$$

Obviously, if $n \cdot l < k$, there are no solutions. Otherwise, we use the Principle of Inclusion and Exclusion to count number of times **exactly** 0 integers violate the criterion using the formula derived above which only counts the cases with **atleast** some number of violations as described below .

Formally, we want to find $|S|$ where $\mathbf{S} = \{\{x_1, x_2 \ldots x_n\} \mid 0 \le x_i < l \; \forall \; 1 \le i \le n\}$ and $\mathbf{U} = \{\{x_1, x_2 \ldots x_n\} \mid 0 \le x_i \le k \; \forall \; 1 \le i \le n\}$. Also let $\mathbf{V_i}$ be the set containing all the sets which have at least i violations; $\mathbf{V_i} = \{\{x_1, x_2 \ldots x_n\}$ and $|A| = i \; s.t. \; A = \{u_1, u_2, \ldots u_i\}, 1 \le u_j \le n, x_{u_j} \ge l \; \forall \; 1 \le j \le i\}$. We note that $V_0 = U$.
Consider the cases where **atleast** i integers : $a_1 \ldots a_i$, $1 \le a_k \le n, 1 \le k \le i$ violate the given constraint. In such a case, we initially dedicate a value of l each of a_i^{th} partition and distribute all the boxes to partitions with $k - i \cdot l$ whilst ensuring that $k - i \cdot l > 0 \implies 0 \le i \le \frac{k}{l}, i \in \mathbb{N} \cup \{0\}$.
The number of such cases when we fix the i integers is $\binom{n+k-i \cdot l-1}{n-1}$. The number of ways to choose i integers is $\binom{n}{i}$. This implies , $|V_i| = \binom{n}{i} \cdot \binom{n+k-i \cdot l-1}{n-1}$.
So, total number of ways is $|S| = |U| - |V_1| + |V_2| ... (-1)^{\lfloor \frac{k}{l} \rfloor} \cdot |V_{\lfloor \frac{k}{l} \rfloor}| = \sum_{0 \le i \le \lfloor \frac{k}{l} \rfloor} (-1)^i \cdot |V_i| = \sum_{0 \le i \le \lfloor \frac{k}{l} \rfloor} (-1)^i \cdot \binom{n}{i} \cdot \binom{n+k-i \cdot l-1}{n-1}$ by the principle of inclusion and exclusion.