CS201

Mathematics For Computer Science

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Question 1

We have seen generating functions for $\binom{n}{m}$ for variable m keeping n fixed, and for variable n keeping m fixed. If we wish to make both variable then the generating function needs to be over two variables.

1. Prove that $\frac{1}{1-y-xy} = \sum_{n\geq 0}\sum_{m\geq 0}\binom{n}{m}x^m y^n$.
2. Derive the generating function $\binom{2n}{n}$ from above two-variable generating function by judicious substitution for one of the two variable.

Solution

1 Part (a) :

The following property can be proved by using the taylor's expansion of it's one variable counterpart. First, let us define another variable z as

$$z = y(1+x) \tag{1.1}$$

Now the given expression can be rewritten equivalently as

$$\frac{1}{1-y-xy} = \frac{1}{1-y(1+x)} \tag{1.2}$$

$$\frac{1}{1-y(1+x)} = \frac{1}{1-z} \tag{1.3}$$

We also know by taylor's expansion of single variable functions that

$$\frac{1}{1-z} = \sum_{n\geq 0} z^n \; for \; |z| \leq 1 \tag{1.4}$$

Therefore using the equations (1.3) and (1.4) we get

$$\frac{1}{1-y-xy} = \sum_{n\geq 0}(y(1+x))^n \; for \; |y(1+x)| < 1 \tag{1.5}$$

$$= \sum_{n\geq 0} y^n(1+x)^n \tag{1.6}$$

Now, using the binomial theorem for $(1+x)^n$ we get

$$\frac{1}{1-y-xy} = \sum_{n\geq 0} y^n \left(\sum_{m\geq 0} \binom{n}{m} x^m \right) \tag{1.7}$$

$$= \sum_{n\geq 0} \left(\sum_{m\geq 0} \binom{n}{m} x^m \right) y^n \tag{1.8}$$

$$= \sum_{n\geq 0} \sum_{m\geq 0} \binom{n}{m} x^m y^n \tag{1.9}$$

Note: We were able to move the terms in parenthesis around as m and n were independent of each other. Also the above relation is only true for $|y(1+x)| < 1$ as only then the sum on the RHS would converge.
Hence, proved that the given relation holds true. □

2 Part (b) :

First, we would be proving some preliminary results that would help us further.

Claim 1: $\sum_{u\geq 0} \binom{2u}{u}\binom{2n-2u}{n-u} = 4^n$

Proof. Let us first define a function, say $f_n(x)$ as

$$f_n(x) = \sum_{i\geq 0} \binom{2i}{i}\binom{2n-2i}{n-i} x^{n-i} \tag{1.10}$$

For proving the claim, we need to find the value of $f_n(1)$, which can be found if we try to find some sort of recursive definition for $f_n(x)$. Now, let's try to define some recursive definition for this function using some observations.

Observation 1: The function $f_n(x)$ satisfies the relation given by

$$f_n(x) = x^n f_n(1/x) \tag{1.11}$$

Proof. The intuition behind defining such a relation comes from the fact that if we substitiute $n-i$ in place of i we do not change the value of binomial coefficient $\binom{n}{i}$. Let us calculate the value of RHS first.

$$x^n f_n(1/x) = x^n \sum_{i\geq 0} \binom{2i}{i}\binom{2n-2i}{n-i} x^{i-n} \tag{1.12}$$

$$\implies x^n f_n(1/x) = \sum_{i\geq 0} \binom{2i}{i}\binom{2n-2i}{n-i} x^{i} \tag{1.13}$$

Now, replacing i with $n-i$ in equation (1.13) gives

$$x^n f_n(1/x) = \sum_{i\geq 0} \binom{2n-2i}{n-i}\binom{2i}{i} x^{n-i} \tag{1.14}$$

$$= f_n(x) \tag{1.15}$$

Hence, proved. □

If we differentiate both sides in observation 1, and plug in $x=1$, we will get another relation for $x=1$.

$$nx^{n-1} f_n(1/x) - x^{n-2} f_n'(1/x) = f_n'(x) \tag{1.16}$$

$$\implies f_n'(1) = n f_n(1) - f_n'(1) \tag{1.17}$$

$$\implies f_n'(1) = \frac{n}{2} f_n(1) \tag{1.18}$$

Now, the natural next step for solving for $f_n(1)$ is to differentiate the original function and try to find another relation to eliminate the derivatives at $x=1$. So, let's proceed with it.

$$f_{n+1}(x) = \sum_{i\geq 0} \binom{2i}{i}\binom{2n-2i+2}{n-i+1} x^{n+1-i} \tag{1.19}$$

$$\implies f'_{n+1}(x) = \sum_{i\geq 0} \binom{2i}{i}\binom{2n-2i+2}{n-i+1} (n+1-i)x^{n-i} \tag{1.20}$$

$$\implies f'_{n+1}(x) = \sum_{i\geq 0} \binom{2i}{i}\binom{2n-2i}{n-i} \left(\frac{(2n-2i+2)(2n-2i+1)}{(n-i+1)(n-i+1)}\right) (n+1-i)x^{n-i} \tag{1.21}$$

$$\implies f'_{n+1}(x) = \sum_{i\geq 0} \binom{2i}{i}\binom{2n-2i}{n-i} (2)(2n-2i+1)x^{n-i} \tag{1.22}$$

$$\implies f'_{n+1}(x) = 2\sum_{i\geq 0} \binom{2i}{i}\binom{2n-2i}{n-i} x^{n-i} + 4\sum_{i=0}^{n+1} \binom{2i}{i}\binom{2n-2i}{n-i} (n-i)x^{n-i} \tag{1.23}$$

$$\implies f'_{n+1}(x) = 2f_n(x) + 4xf'_n(x) \tag{1.24}$$

$$\implies f'_{n+1}(1) = 2f_n(1) + 4f'_n(1) \tag{1.25}$$

Using the relation between value of differential at $x = 1$ obtained for general n in eq. (1.18), we can substitute them in eq. (1.25) to get a nice recursive relation which we desire for.

$$\left(\frac{n+1}{2}\right) f_{n+1}(1) = 2nf_n(1) + 2f_n(1) \tag{1.26}$$

$$\implies f_{n+1}(1) = 4f_n(1) \tag{1.27}$$

Therefore, we got the final recursive relation for value of function at $x = 1$. Now, observe the fact that the value of $f_0(1)$ is 1. Hence, $f_n(1)$ is a geometric progression with common ratio 4 and initial term 1, which implies the general value of $f_n(1)$ is 4^n. Hence, proved. $\square$

Moving further, if we plug in the value of variable x to be 3 in the equation of Part (a) we get

$$\frac{1}{1-4y} = \sum_{n\geq 0}\sum_{m\geq 0} \binom{n}{m} 3^m y^n \tag{1.28}$$

$$= \sum_{n\geq 0} y^n (1+3)^n \tag{1.29}$$

$$= \sum_{n \geq 0} 4^n y^n \tag{1.30}$$

Let $f(y)$ be the generative function for $\binom{2n}{n}$, therefore

$$f(y) = \sum_{u \geq 0} \binom{2u}{u} y^u \tag{1.31}$$

$$\implies f^2(y) = \sum_{u \geq 0} \sum_{v \geq 0} \binom{2u}{u} \binom{2v}{v} y^{(u+v)} \tag{1.32}$$

$$Let\ u + v = n \tag{1.33}$$

$$\implies f^2(y) = \sum_{n \geq 0} \sum_{u \geq 0} \binom{2u}{u} \binom{2n-2u}{n-u} y^n \tag{1.34}$$

$$\tag{1.35}$$

Now, by using **Claim 1**, we get

$$f^2(y) = \sum_{n \geq 0} 4^n y^n \tag{1.36}$$

$$\implies f^2(y) = \frac{1}{1-4y}\ using\ (1.18) \tag{1.37}$$

$$\implies f(y) = \pm(1-4y)^{-1/2} \tag{1.38}$$

Now, we can easily see that amongst the two possible $f(y)$ obtained in the equation (1.26) only one is acceptable as one of them is strictly the negative of the other and hence the sequence represented by the it is just the negative of the sequence obtained using the other. Therefore, we need to select only one out of these two and this can easily be checked by plugging in $y = 0$ in the taylor expansion of both of these. Since, $f(0)$ should be positive number, hence $(1 - 4y)^{-1/2}$ is the only acceptable solution amongst the two. Therefore,

$$f(y) = (1-4y)^{-1/2}$$

Question 2

For a fixed number $k > 0$, find the recurrence relation and generating function for the sequence $a_n^k = \lfloor \frac{n}{k} \rfloor$. Use these two to derive the generating function for the sequence $b_n^k = \left(\lfloor \frac{n}{k} \rfloor \right)^2$.

Solution

Assuming that $\mathbf{n}$ and $\mathbf{k}$ both are natural numbers.
For the recurrence relation, we observe that :

$$a_{n+k}^k - a_n^k = \lfloor \frac{n+k}{k} \rfloor - \lfloor \frac{n}{k} \rfloor = 1$$

. Also, $a_n^k = 0$ for all $n < k$ as $\lfloor \frac{n}{k} \rfloor = 0$ whenever $n < k$.
The generating function $f_k(x)$ looks like:

$$f_k(x) = 0 + 0x + 0x^2 + 0x^3 \dots 0x^{k-1} + 1x^k + 1x^{k+1} \ \dots \ 1x^{2k-1} + 2x^{2k} \dots$$

. Each integer q occurs k times as coefficient of terms $x^{kq}, x^{kq+1} \dots x^{kq+k-1}$. Taking x^{kq} as common from these terms we obtain,

$$f_k(x) = 0x^{k0}(1 + x + x^2 \dots x^{k-1}) + 1x^k(1 + x + x^2 \dots x^{k-1}) \dots qx^{kq}(1 + x + x^2 \dots x^{k-1})$$

$$\implies f_k(x) = (1 + x + x^2 \dots x^{k-1})(\sum_{i \geq 0} ix^{ki}) = \frac{1 - x^k}{1 - x} \cdot \sum_{i \geq 0} ix^{ki}$$

.
Let $G(x)$ be a generating function such that ,

$$G(x) = \sum_{i \geq 0} ix^i$$

. For $|x| < 1$, we have $\frac{1}{1-x} = \sum\limits_{i \geq 0} x^i$.
Differentiating it w.r.t. x and multiplying by x, we get

$$\frac{x}{(1-x)^2} = \sum_{i \geq 0} ix^i = G(x)$$

$$\implies G(x^k) = \frac{x^k}{(1-x^k)^2} = \sum_{i\geq 0} i x^{ki}$$

as required for $f_k(x)$ above.
So,

$$f(x) = \frac{1-x^k}{1-x} \cdot \frac{x^k}{(1-x^k)^2} = \frac{x^k}{(1-x)(1-x^k)}$$

. This is required generating function for a_n^k.

Similarly for $b_n^k = (\lfloor \frac{n}{k} \rfloor)^2$, the generating function

$$f_k^*(x) = \frac{1-x^k}{1-x} \cdot \sum_{i\geq 0} i^2 x^{ki} = \frac{1-x^k}{1-x} \cdot G^*(x^k)$$

say.
So, we differentiate both sides of the equation

$$\frac{x}{(1-x)^2} = \sum_{i\geq 0} i x^i = \frac{x-1+1}{(1-x)^2} = \frac{1}{(1-x)^2} - \frac{1}{1-x}$$

to get:

$$\frac{2}{(1-x)^3} - \frac{1}{(1-x)^2} = \frac{x+1}{(1-x)^3}$$

. Multiplying by x,

$$\sum_{i\geq 0} i^2 x^i = \frac{x(x+1)}{(1-x)^3} = G^*(x)$$

. Then,

$$f_k^*(x) = \frac{1-x^k}{1-x} \cdot G^*(x^k) = \frac{1-x^k}{1-x} \cdot \frac{(x^k)(x^k+1)}{(1-x^k)^3} = \frac{(x^k)(x^k+1)}{(1-x)(1-x^k)^2}$$

. This is the required generating function for b_n^k.

Question 3

Given numbers from 0 to $2n - 1$ in a sequence, what is the number of permutations of this sequence such that no even number is in its original position (express the number of permutations in terms of derangement number d_n)?

Solution

Derangement: d_n is the number of ways we can permute n objects in such a way that, none of the objects occupy their original position.

Let P denote the set of all permutations of $[0, 2n - 1]$. Define $A_k \subset P$ s.t. $\forall\, a \in A_k$ where a is a permutation, then $n - k$ odd numbers in a are in their original place while the other k odd numbers and all n even numbers are not in their original place for $k \in [0, n]$.
$\{A_k\}\ \forall\, k \in [0, n]$ are **disjoint** :
For $k, l \in [0, n], A_k \cap A_l = \phi$ as if $a \in A_k$, then $n - k$ odd numbers in a are in their original place and others are in different place, so total number of odd integers in original place in a can never be more or less than $n - k$ and hence $a \notin A_l\ \forall\, l \in [0, n]/k$.

Let set of permutations in which even number are not in their original place, $\mathbf{A_E} \subset \mathbf{P}$.

A_E = (set of permutations where no number is in its original place) $\bigcup$ (set of permutations where one odd number is in its original place and others not in original place) $\bigcup \ldots$ (set of permutations where all odd numbers are in their original place and all other even numbers are not in original place)

$$\begin{aligned} A_E &= \bigcup_{k \in [0,n]} A_k \\ |A_E| &= \left| \bigcup_{k \in [0,n]} A_k \right| \\ &= \sum_{k \in [0,n]} |A_k| \quad (A_k \cup A_l = \phi, \forall k, l \in [0, , n]) \end{aligned}$$

For each A_k:
Let the $2n$ numbers be broken into **2** subgroups where the first subgroup is to be **de-arranged** and the second subgroup occupies their **original** position as before.

For the first subgroup, we must take all the $\mathbf{n}$ even numbers and $\mathbf{k}$ odd numbers that will **not** occupy their original position, by $\binom{n}{k}$ ways.Then the rest $n-k$ odd numbers must occupy their original position in exactly 1 way. We can derange the first group by d_{n+k} ways. For a particular value of k, we have $d_{n+k}\binom{n}{k}$ permutations.

So, the total number of all such permutations will be sum of cardinalities of all these sets which is $\mathbf{A_E} = \sum_{\mathbf{k=0}}^{\mathbf{n}} \mathbf{d_{n+k}}\binom{\mathbf{n}}{\mathbf{k}}$.

Question 4

Let A be a set containing non-empty sets and define $A_\times = \prod_{B\in A} B$. Prove that Axiom of Choice is equivalent to the statement that for every set A as above, $A_\times \neq \emptyset$.

Solution

Proof of Axiom of Choice implies the statement in question.
Consider an arbitrary set A s.t. $\phi \notin A$
By Axiom of Choice,

$$\exists f : A \to \bigcup_{B\in A} B \text{ s.t. } f(B) \in B \,\forall\, B \ \in A$$

By definition of cross product,

$$\exists \text{ set } I \text{ s.t. } |I| = |A| \Rightarrow \exists g : I \to A \text{ s.t. } g \text{ is a bijection, then}$$

$$A_X = \prod_{B\in A} B = \{\{b_\alpha | b_\alpha \in g(\alpha), \alpha \in I\}\}$$

Consider the tuple $a = \{f(g(\alpha))|\alpha \in I\}$. Since $f(g(\alpha)) \in g(\alpha),$ for $\alpha \in I$ by definition of f,

$$\Rightarrow \{f(g(\alpha))|\alpha \in I\} \in \{\{b_\alpha | b_\alpha \in g(\alpha), \alpha \in I\}\}$$

$$\Rightarrow \{f(g(\alpha))|\alpha \in I\} \in A_X$$

$$\therefore A_X \neq \phi$$

Since we have proved $A_X \notin \phi$ for arbitrary A s.t. $\phi \notin A$, it implies result holds for all such sets. □

Proof of the statement in question implies Axiom of Choice.
Consider arbitrary set A s.t. $\phi \notin A$. By definition of cross product,

$$\exists \text{ set } I \text{ s.t. } |I| = |A| \Rightarrow \exists g : I \to A \text{ s.t. } g \text{ is a bijection, then}$$

$$A_X = \prod_{B\in A} B = \{\{b_\alpha | b_\alpha \in g(\alpha), \alpha \in I\}\}$$

By statement in question, $A_X \neq \phi \Rightarrow \exists a \in A_X$.

$$\Rightarrow a = \{b_\alpha | \alpha \in I\} \text{ for some } b_\alpha \in g(\alpha), \text{ for } \alpha \in I$$

Since g is a bijective function, $g^{-1} : A \to I$ exists. Now constructing function $f : A \to \bigcup_{B \in A} B$ as:

$$f(B) = b_{g^{-1}(B)}$$

Since $b_\alpha \in g(\alpha) \therefore f(B) = b_{g^{-1}(B)} \in g(g^{-1}(B)), \forall B \in A$.

$$\Rightarrow f(B) \in B, \forall B \in A \quad as\ (g(g^{-1}(x)) = x)$$

$\therefore \exists f : A \to \bigcup_{B \in A} B,$ s.t. $f(B) \in B \; \forall \; B \in A$, for arbitrary A s.t. $\phi \notin A$, hence the result holds for all such sets. $\square$

Since the statement in question and Axiom of Choice both imply each other, they are equivalent.