CS203B - PROBABILITY FOR COMPUTER SCIENCE
NITIN SAXENA

ASSIGNMENT 3

POINTS: 35

DATE GIVEN: 11-APR-2021 DUE: 26-APR-2021

Rules:
- You are strongly encouraged to work *independently*. That is the best way to understand the subject.
- Write the solutions on your own and honorably *acknowledge* the sources if any. `cse.iitk.ac.in/pages/AntiCheatingPolicy.html`
- Clearly express the fundamental *idea* of your proof/ algorithm before going into the other proof details. The distribution of partial marks is according to the proof steps.
- There will be a penalty if you write unnecessary or unrelated details in your solution. Also, do not repeat the proofs done in the class.
- Submit your solutions, before time, to your TA via MooKit. Preferably, submit a printed/pdf copy of your LaTeXed or Word processed solution sheet.
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Question 1: *[2+7 points]* Suppose there is a town with n persons. Each one of them is lazy but quite fond of spreading fake news. On some day, a person hears some fake news. The next morning, s/he uniformly randomly picks a phone number from the telephone directory, and communicates the fake news to the person with that number. So, now there are 2 persons who know the fake news. As you can imagine, the fake news starts spreading according to the following protocol.

Every morning, each person who knows the fake news, picks a phone number uniformly randomly from the telephone directory, and communicates the fake news to the other person.

You may observe that there is a possibility that a person calls the same person multiple times. Moreover, a single person may get a call from multiple persons on the same day. Hence, every phone call may not necessarily increase the number of persons knowing the fake news.

Let X be a random variable for the number of days it takes for each person of the town to know the fake news. It can be seen that for every m, however large it may be, $P(X > m)$ is non-zero.

Show that $\log_2 n$ is the least number of days to spread the fake news to the entire town. I.e. $P(X < \log_2 n) = 0$.

What is the expected number of days $E[X]$ it will take to spread the fake news? Give a 'nice' expression and a convincing heuristic analysis (as you may find the rigorous proof too difficult).

Question 2: *[9 points]* On a long straight road lies a house at 0-th milestone and a bar at n-th milestone. There is a drunkard standing at i-th milestone. He decides to walk the next mile towards the bar or the home with equal probability. He repeats this on reaching each milestone. The decision (to walk the next mile towards home or bar) at each milestone is independent of all the past moves (due to the large amounts of Vodka he had). The walk stops as soon as the drunkard reaches home or bar.

Let $e(i)$ be the expected number of steps in which the walk stops starting from the i-th milestone. Let $p(i)$ be the probability that the drunkard starting from the i-th milestone reaches home.

Fix $n = 4$ for an easier calculation. Find $e(i)$ and $p(i)$, $i \in [3]$.

(Optional question) What are these values for general n?

Question 3: *[3+6 points]* When an event E happens, you often feel an emotion of 'surprise'. Let us try to measure this emotion mathematically. Assume that the surprise depends only on the probability $p := P(E)$, that the mind has somehow estimated. Denote the *surprise* by a non-negative function $S(p)$. It is defined only for p in the range $[0, 1]$. Unsurprisingly, the surprise function should satisfy the following natural axioms.

Axiom 1. $S(1) = 0$.

Since there is *no* surprise when an event E happens, that has probability $p = 1$.

Axiom 2. $p > q \Rightarrow S(p) < S(q)$.

Since the surprise on event-1 happening is *less*, if its probability is known to be larger than event-2.

Axiom 3. $S(p)$ is a *continuous* function of p.
Axiom 4. $S(pq) = S(p) + S(q)$. Why is this justified?

Show that the surprise function $S(p)$ is given by $S(1/2) \cdot \log_2(1/p) \geq 0$, for $p \in [0, 1]$.

It is customary to define $S(1/2) := 1$; making $S(p) = \log_2(1/p)$.

Question 4: *[8 points]* Suppose that every vertex of an n-vertex *bipartite* graph is given a personalized list of $> \log_2 n$ possible colors. Prove that it is possible to give each vertex a color, from its list, such that no two adjacent vertices receive the same color.

[Hint: Why is this a probability question?]

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