CS203: Probability in Computer Science
Assignment 1 Solutions

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I Problem 1 Solution

We were given two dices which were rolled together and gave S_0 and S_1 as outcomes. We were asked to find the probability that the quadratic equation $x^2 + S_1 x + S_0 = 0$ has real roots.

We know from the property of quadratic equations that for it to have real roots, the discriminant should be greater than or equal to zero. Hence, we need to find solutions that satisfy the equation,

$$S_1^2 \geq 4S_0 \tag{I1}$$

We will also assume that the set of dices that are rolled are indistinguishable as otherwise there would be no way to choose the values S_1 and S_0 from the rolled outcomes. Now, we will iterate over the favourable outcomes that satisfy the equation (I1) in the following way. Fix the outcome S_0 and iterate over the values of S_1 that satisfy (I1).

Case 1: $S_0 = 1 \rightarrow$ The favourable options for S_1 would be $\{2,3,4,5,6\}$ which yields the set $X_1 = \{\{1,2\},\{1,3\},\{1,4\},\{1,5\},\{1,6\}\}$ containing ordered pairs.

Case 2: $S_0 = 2 \rightarrow$ The favourable options for S_1 would be $\{3,4,5,6\}$ which yields the set $X_2 = \{\{2,3\},\{2,4\},\{2,5\},\{2,6\}\}$ containing ordered pairs.

Case 3: $S_0 = 3 \rightarrow$ The favourable options for S_1 would be $\{4,5,6\}$ which yields the set $X_3 = \{\{3,4\},\{3,5\},\{3,6\}\}$ containing ordered pairs.

Case 4: $S_0 = 4 \rightarrow$ The favourable options for S_1 would be $\{4,5,6\}$ which yields the set $X_4 = \{\{4,4\},\{4,5\},\{4,6\}\}$ containing ordered pairs.

Case 5: $S_0 = 5 \rightarrow$ The favourable options for S_1 would be $\{5,6\}$ which yields the set $X_5 = \{\{5,5\},\{5,6\}\}$ containing ordered pairs.

Case 6: $S_0 = 6 \rightarrow$ The favourable options for S_1 would be $\{5,6\}$ which yields the set $X_3 = \{\{6,5\},\{6,6\}\}$ containing ordered pairs.

Therefore, the favourable cases would be pairs in the set $X = \bigcup_{i=1}^{6} X_i$. Since, all X_is are disjoint, the cardinality of X would just be sum of cardinalities of X_is, which is 19. Therefore, the probability comes out to be $19/(6*6) = 19/36$.

II Problem 2 Solution

In this problem, we were asked to find the probability of choosing a red ball after choosing some number of balls from the bag. We can solve this problem easily using the Partition formula.

Consider the set of events

$$B_i : i\text{th face appears.}$$

Note that the event of choosing some number of balls from the urn is partitioned by event B_is with i running from 1 to n and all of them are trivially disjoint.

We can therefore use this partition to calculate the required probability as

$$P(\text{getting a red ball}) = \sum_{i=1}^{n} P(B_i) \cdot P(\text{getting a red ball} \mid B_i)$$

Now, since the die is fair, there each of the B_i occurs with uniform probability. Hence, $P(B_i) = 1/n$. Now, what is left is the other probability. For calculation of the second probability, consider another set of events

$$A_{j,r} : \text{Event that } j \text{ of the balls are red given that } r \text{ balls selected.}$$

Note that the sets $A_{j,r}$ are disjoint and also cover the complete event B_i with j running from 0 to r, which can be seen trivially. Now, we will use this partition to calculate the second probability.

$$P(\text{getting a red ball} \mid B_r) = \sum_{j=0}^{r} P(A_{j,r}) \cdot P(\text{getting a red ball} \mid A_{j,r})$$

Now, for calculating $P(A_{j,r})$, we do the following

$$\begin{aligned} P(A_{j,r}) = P(j \text{ are red} \mid r \text{ were selected}) &= \frac{P(j \text{ are red} \cap r \text{ were selected})}{P(r \text{ were selected})} \\ &= \frac{\frac{1}{2^j} \cdot \frac{1}{2^{r-j}}}{\sum_{k=0}^{r} \frac{1}{2^k} \cdot \frac{1}{2^{r-k}}} \\ &= \frac{1}{r+1} \end{aligned}$$

Also, the second probability can be calculated directly as well, since choosing all the balls are equally likely. Therefore, $P(\text{getting red ball} \mid A_{j,r}) = j/r$. Substituting this above, we get

$$\begin{aligned}
P(\text{getting a red ball} \mid B_r) &= \sum_{j=0}^{r} \frac{1}{r+1} \cdot \frac{j}{r} \\
&= \frac{1}{r \cdot (r+1)} \sum_{j=0}^{r} j \\
&= \frac{r \cdot (r+1)}{2 \cdot r \cdot (r+1)} \\
&= \frac{1}{2}
\end{aligned}$$

Therefore, plugging this in original expression, we get

$$\begin{aligned}
P(\text{getting a red ball}) &= \sum_{i=1}^{n} P(B_i) \cdot P(\text{getting a red ball} \mid B_i) \\
&= \sum_{i=1}^{n} \frac{1}{n} \cdot \frac{1}{2} \\
&= \frac{1}{2}
\end{aligned}$$

Hence, the final probability comes out to be $\frac{1}{2}$.
Note: This probability can be intuitively obtained considering the symmetry present in the original problem. However, the above provides a rigorous argument for the same.

III Problem 3 Solution

In this problem we were required to show that any finite sequence of length r appears in the infinite coin toss. We can prove this if we can prove that the probability of a certain sequence appearing is 1 in the infinite toss model. For this we will first consider a finite length of coin tosses and find the probability of this sequence of length r being present in it.

Since, it is difficult to find the probability of appearing of sequence, we will try to find the probability by which this sequence never appears in this finite coin toss. For this, consider n coin flips first.

Claim 3.0.1. *For coin tosses of length n, the probability that a given sequence never appears is upper bounded by the probability that it never appears in any of the* $\lfloor \frac{n}{r} \rfloor$ *ranges* $r(i-1)+1 \ldots ri$ *for* $1 \leq i \leq \lfloor \frac{n}{r} \rfloor$.

Proof. The proof for the last statement can be seen as we are only looking at non-overlapping ranges and while adding favourable cases in which these non-overlapping ranges do not have the sequence, we might encounter certain cases where the sequence appears in the other r length ranges besides the one discussed above. Hence, those cases actually need to be subtracted from the currently counted ones and hence, the probability found using this acts as an upper bound to the required probability. □

The probability of the sequence never appearing in the $\lfloor \frac{n}{r} \rfloor$ ranges can be multiplied together to obtain the required probability of sequence not appearing in the whole range since all the ranges are disjoint and hence independent from each other. The probability for one such range will be $P_0 = (1 - \frac{1}{2^m})$, which is essentially the complement of the probability of that sequence occurring in the range. Therefore, the total probability would be $P_0^{\lfloor \frac{n}{r} \rfloor}$. Let the actual probability be P. Therefore, using **Claim 3.0.1** we know that

$$P \leq P_0^{\lfloor \frac{n}{r} \rfloor}$$

Hence, the probability of the sequence occurring in the coin toss would just be $P_{req} = 1 - P$.

$$P \leq P_0^{\lfloor \frac{n}{r} \rfloor} \tag{III1}$$

$$P_{req} \geq 1 - P_0^{\lfloor \frac{n}{r} \rfloor} \tag{III2}$$

Now, as $n \to \infty$, since $P_0 = (1 - \frac{1}{2^m}) \leq 1$, therefore $P_0^{\lfloor \frac{n}{r} \rfloor} \to 0$. Hence, using (III2), we get that the required probability tends to 1. Hence, proved.

IV Problem 4 solution

Since the problem was a bit controversial based on whether the envelops are to be assumed same (as given in forum) or distinct (as stated in question paper, that they are present in a line), I would be providing solutions to both the cases.
Case (a) → Assuming envelops to be identical:
The problem in hand is placing the letters into envelops such that no letter goes into the envelop of it's own color. Assuming the letters and envelops are to be identical we will enumerate the total cases and favourable cases and then calculate the probability. First let's count the favourable cases. These will occur in two types.

Case 1: In this case, two same colored letters will go to the same colored envelopes. For the first pair of same colored letters there will be two ways of choosing the pair of envelops (i.e. the other two colors). After fixing the position of one pair, the other two gets fixed. Therefore, total number of favourable cases would be 2 corresponding to this case.

Case 2: In this case, none of the pairs of letters of same color go into envelops of same color. Therefore, red will go into blue and white envelops, blue will go into red and white envelops and white into red and blue envelops. Therefore, this will add 1 more favourable case.

Therefore, total number of favourable cases will be 3. Now, what is left is to calculate the favourable cases. We will partition those into 4 categories.

Case 1: In this case, all letters of same color go into envelops of same color. If this happens to be the case then we just need to permute these pairs which leads to 3! or 6 cases.

Case 2: In this case, two pairs of letters go into envelops of same color. If this happens, the third pair would also have to go into the same colored envelops, which we have already counted above. Hence this accounts for 0 cases.

Case 3: In this case, only one pair of letters go into envelops of same color. If this happens, then there are three options to select this pair and three options to select positions for this pair and the rest of the 4 letters will be placed in one way. Therefore, this gives rise to 9 new cases.

Case 4: In this case, none of the pairs of same colored letters go into envelops of same color. This means we have to permute the pairs of the type {RW, WB, RB} which can be done in 3! or 6 ways.

Hence, the total number of favourable cases are $6+0+9+6=21$. And therefore the required probability comes out to be 1/7.

Case (b) → Assuming envelops to be distinct:
The problem in hand is placing the letters into envelops such that no letter goes into the envelop of it's own color. First we will solve the variant of the problem assuming that the two letters (and envelops) of the same color are distinct in nature and then we can see that probability obtained will be the same since both the number of favourable cases and total cases will reduce by a factor of 8 because of the fact that permutations of two letters of same color will be counted as one and since there are 3 pairs, therefore we will divide total cases by 2^3. Therefore, we can reformulate the given problem into the the following equivalent problem - Find the number of bijective mapping from the set of natural numbers in range $[1,6]$ to the natural numbers in range $[1,6]$ with the following constraints

1. 1 should not map to 1 or 2.
2. 2 should not map to 1 or 2.
3. 3 should not map to 3 or 4.
4. 4 should not map to 3 or 4.
5. 5 should not map to 5 or 6.
6. 6 should not map to 5 or 6.

Let us define the events A_i in the following way

$$A_i \rightarrow \text{Event such that constraint number } i \text{ is satisfied}$$

For eg. - A_1 is the event such that 1 is not mapped to 1 or 2.
We are asked to find the probability of the occurance of the event $\bigcap_{i=1}^{6} A_i$. Now, we will use the generalised form of De-Morgan's law and let Ω be the universal set of all the bijective mappings from the range $[1-6]$ to the range $[1-6]$.

$$\bigcap_{i=1}^{6} A_i = \overline{\bigcup_{i=1}^{6} \overline{A_i}} \tag{IV1}$$

$$\implies \left|\bigcap_{i=1}^{6} A_i\right| = |\Omega| - \left|\bigcup_{i=1}^{6} \overline{A_i}\right| \text{ where } |X| \text{ denotes the cardinality of the set X} \tag{IV2}$$

Now, using (IV2), we get that we can find $\left|\bigcup_{i=1}^{6} \overline{A_i}\right|$ instead and obtain the required result from that. Now, this can be easily found using the inclusion-exclusion principle. I will list each of the terms that would appear while using inclusion-exclusion principle and find their values.

1. The cardinalities of single sets ($|\overline{A_i}|$) $\rightarrow$ This would be the event when constraint i is not satisfied or i maps to one of the two values it is not supposed to map to. This can occur in 2 ways and rest of the numbers can be mapped in 5! ways. There will be 6 such individual sets, hence this term will give rise to $6 \cdot (2 \cdot 5!)$ possibilities.

2. The cardinalities of intersection of two sets ($|\overline{A_i} \cap \overline{A_j}|$) $\rightarrow$ These types of intersections would be in two types, 3 intersections would be of type when i and j are supposed to match within i and j. For eg - $\overline{A_1}$ and $\overline{A_2}$. This type will give rise to 2! matches within itself (i and j mapping within i and j only) and 4! for the rest of the numbers. Therefore it will give $3 \cdot (2! \cdot 4!)$ cases. The second possibility is the other scenario, which is easier to handle and gives rise to $\binom{6}{2} - 3$ pairs and each will give rise to $2 \cdot 2 \cdot 4!$ cases. The first 2 for selection of the mapping of i, the second for the mapping of j and other elements can be mapped arbitrarily then. This given rise to total of $\left(\binom{6}{2} - 3\right) \cdot (2 \cdot 2 \cdot 4!)$ cases.

3. The cardinalities of intersection of three sets ($|\overline{A_i} \cap \overline{A_j} \cap \overline{A_k}|$) $\rightarrow$ These types of intersections would be in two types, some intersections would be of type when i and j are supposed to match within i and j and the k maps to some other value. For eg - $\overline{A_1}$ and $\overline{A_2}$ and $\overline{A_5}$. There are $\binom{3}{2}$ ways of selecting the indexes of type i, j and then 4 ways of choosing k. This type will give rise to 2! matches within itself (i and j mapping within i and j only) and 2! for k and 3! for the rest of the numbers. Therefore it will give $\binom{3}{2} \cdot 4 \cdot (2! \cdot 2! \cdot 3!)$ cases. The second possibility is the other scenario when i, j and k, all come from different pairs, which is easier to handle and gives rise to $\binom{6}{3} - \binom{3}{2} \cdot 4$ tuples of events and each will give rise to $2 \cdot 2 \cdot 2 \cdot 3!$ cases. The first 2 for selection of the mapping of i, the second for the mapping of j, third for the mapping of k and other elements can be mapped arbitrarily then. This given rise to total of $\left(\binom{6}{3} - \binom{3}{2} \cdot 4\right) \cdot (2 \cdot 2 \cdot 2 \cdot 3!)$ cases.

4. The cardinalities of intersection of four sets ($|\overline{A_i} \cap \overline{A_j} \cap \overline{A_k} \cap \overline{A_l}|$) $\rightarrow$ These types of intersections would be in two types, some intersections would be of type when i and j are supposed to match within i and j and the k and l map within k and l. For eg - $\overline{A_1}$ and $\overline{A_2}$ and $\overline{A_3}$ and $\overline{A_4}$. There are $\binom{3}{2}$ ways of selecting the indexes of type i, j and k, l. This type will give rise to 2! matches

within itself (i and j mapping within i and j only) and 2! for k and l and 2! for the rest of the numbers. Therefore it will give $\binom{3}{2}\cdot(2!\cdot 2!\cdot 2!)$ cases. The second possibility is the other scenario when i, j map within themselves and k and l come from different pairs, which gives rise to $\binom{6}{4}-\binom{3}{2}$ tuples of events and each will give rise to $2\cdot 2\cdot 2\cdot 2!$ cases. The first 2! for selection of the mapping of i,j, the second for the mapping of k, third for the mapping of l and other elements can be mapped arbitrarily then. This given rise to total of $\left(\binom{6}{4}-\binom{3}{2}\right)\cdot(2\cdot 2\cdot 2\cdot 2!)$ cases.

5. The cardinalities of intersection of 5 sets ($|\overline{A_i}\cap\overline{A_j}\cap\overline{A_k}\cap\overline{A_l}\cap\overline{A_m}|$) $\rightarrow$ This gives rise to $\binom{6}{5}$ tuples of events and each will give rise to $2\cdot 2\cdot 2$ cases. The first 2! for selection of the mapping of i,j, the second for the mapping of k,l, third for the mapping of l, the other element's mapping gets fixed due to this. This given rise to total of $\binom{6}{5}\cdot(2\cdot 2\cdot 2)$ cases.

6. The cardinality of intersection of all 6 sets $\rightarrow$ This would just be equal to $2!\cdot 2!\cdot 2!$ that is arrangement within the three pairs.

Now, applying inclusion-exclusion principle and plugging in values from the above enumeration, we get

$$\begin{aligned}\left|\bigcup_{i=1}^{6}\overline{A_i}\right| &= \text{The cardinalities of single sets} - \text{The cardinalities of intersection of two sets}\\ &+ \text{Cardinalities of intersection of three sets} - \text{Cardinalities of intersection of four sets}\\ &+ \text{Cardinalities of intersection of five sets} - \text{Cardinalities of intersection of six sets}\\ &= 6\cdot(2\cdot 5!) - 3\cdot(2!\cdot 4!) - \left(\binom{6}{2}-3\right)\cdot(2\cdot 2\cdot 4!) + \binom{3}{2}\cdot 4\cdot(2!\cdot 2!\cdot 3!) + \left(\binom{6}{3}-\binom{3}{2}\cdot 4\right)\cdot(2\cdot 2\cdot 2\cdot 3!)\\ &- \binom{3}{2}\cdot(2!\cdot 2!\cdot 2!) - \left(\binom{6}{4}-\binom{3}{2}\right)\cdot(2\cdot 2\cdot 2\cdot 2!) + \binom{6}{5}\cdot(2\cdot 2\cdot 2) - 2!\cdot 2!\cdot 2! = 640\end{aligned}$$

Also, we know that $|\Omega| = 6! = 720$. Therefore, using (IV2), we get

$$\begin{aligned}\left|\bigcap_{i=1}^{6}A_i\right| &= |\Omega| - \left|\bigcup_{i=1}^{6}\overline{A_i}\right|\\ &= 720 - 640\\ &= 80\end{aligned}$$

Hence, the probability of event $\bigcap_{i=1}^{6} A_i$ will be $\frac{80}{|\Omega|}$ = 1/9. This will be same even if two letters(and envelops) of same color are identical.

4.1 Problem 5 solution

The problem required us to give an algorithm for the hiring process of BigBucks. Firstly, we will analyse the problem in hand.

It is trivial to see that what matters to order the candidates is just the order of the qualities of the candidates and not the actual value of quality, hence we can forget about the distribution from which the qualities will come from totally.

Also, notice that we will never accept a candidate with quality score less than one we have rejected earlier. So, now we only need to analyse candidates with quality higher than the previous ones. Let's call them as 'good' candidates.

Now, consider the first candidate he will always be a 'good' candidate. But the probability that the best candidate is amongst the remaining candidates is quite high in this case. Similarly, consider the next 'good' candidate. The probability that the best candidate is amongst the remaining ones is lesser than what we got for the first candidate since there are fewer candidates left to be seen. Therefore, we can say that the probability of best candidate being in the remaining section decreases and therefore the probability that current hopeful is the best one increases. The two options available at each 'good' candidate is either to accept or reject him. Therefore, the probability of picking the best candidate by accepting the current 'good' candidate increases with the number of candidates seen so far and the probability of picking the best candidate by rejecting the current 'good' candidate decreases with the number of candidates seen so far. The moment at which first probability is greater than the second accept the candidate, otherwise reject the candidate.

Now, for further analysis we would be in requirement of finding out the probabilities. Notice, that initially probability for rejecting the current 'good' candidate should be higher and it should decrease afterwards and the first probability will keep increasing. Hence, after a certain stage the first probability will become higher than the second and we should then accept the next 'good' candidate after this stage as the best one. This would be the ideal strategy.

Let's find out the probabilities now. Denote by $P(r)$, the probability of selecting the best candidate after rejecting the r initial ones. Now, this probability can be partitioned with the help of the position at which the best candidate is. Note that all of these sets would be trivially disjoint. Therefore, we write

$$P(r) = \sum_{i=r+1}^{n} P(\text{selecting best candidate being rejected first r candidates} \mid \text{Best is at } i^{th}\text{position}) \cdot P(\text{Best is at } i^{th}\text{position})$$

Since the order of candidates appearing is random therefore the best candidate can occur at any position uniformly. Hence, $P(\text{Best is at } i^{th}\text{position}) = 1/n$.
For calculating the other part, we need to ensure that the best candidate is the first 'good' candidate after rejecting r candidates. For this to be true, the best among the initial $(i-1)$ candidates lies within the first r candidates. Since, the order of candidates is random, this happens with probability $\frac{r}{i-1}$. Thus,

$$P(\text{selecting best candidate being rejected first r candidates} \mid \text{Best is at } i^{th}\text{position}) = \frac{r}{i-1}$$

Therefore, from the above expression of $P(r)$ in terms of partition, we have

$$P(r) = \frac{r}{n} \sum_{i=r+1}^{n} \frac{1}{i-1}$$

The problem for finding the best candidate with highest probability boils down to finding a suitable r for which $P(r)$ is maximum. This helps us deriving our BigBucks' hiring algorithm.

Algorithm 1: BigBucks' Hiring Algorithm

```
Function P(int r, int n):
    sum ← 0
    for i in range r+1...n do
        sum += 1/(i-1)
    return r/n · sum
Function get_optimal_r(int n):
    max ← 0
    max_ind ← 0
    for i in range 1...n do
        if P(i,n) ≥ max then
            max_ind ←i
            max ←P(i)
    return max_ind
```

Algorithm 2: BigBucks' Hiring Algorithm

Input: Array A containing list of candidates
Output: Best candidate

```
Function get_candidate(Array A, int n):
    optimal_r ← get_optimal_r(n)
    max_till_r ← 0
    best ← 0
    for i in range 1...r do
        quality ← get_quality(A[i])
        if quality ≥ max_till_r then
            max_till_r ← quality
    for i in range r+1...n do
        quality ← get_quality(A[i])
        if quality ≥ max_till_r then
            best ← i
            break
    return best
```

The above algorithm will not work correctly when $r = 0$, but in that case, always the first candidate would be selected and hence $P(r) = 1/n$ and now, this can be compared with the other probabilities of different rs. The exact analytical solution for the value of $P(r)$ can be found when n tends to infinity as then we can convert this summation into definite integral as follows by letting r/n as the variable x, we get

$$\begin{aligned} P(x) &= x \cdot \int_x^1 1/y \, dy \\ &= -x \cdot \ln(x) \end{aligned}$$

Now, we can maximise this analytical value with respect to x. Since, the second derivative is always negative for positive x, therefore the positive stationary point would be a maxima and it comes at $x = 1/e$. Hence, the optimal r for large value of n tends to n/e, which gives the approximate probability of selecting the best candidate for large n as $1/e$ or 36.78%. [1]

References

[1] Secretary Problem. https://en.wikipedia.org/wiki/Secretary_problem.