CS203: Probability in Computer Science
Assignment 2 Solutions

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I Problem 1 Solution

In this problem, we were asked to find the probability of the longest sequence of consecutive heads being not more than $2\log n$. We will solve the complement of this problem first i.e. what is the probability of the length of longest sequence of consecutive heads being more than $2\log n$. For this let us define some events first:

$$A \rightarrow \text{Length of longest sequence of consecutive heads is more than } 2\log n,$$
$$A_i \rightarrow \text{Event that tosses in range } [i, i + 2logn - 1] \text{ are all heads}$$

We can see that all the favourable outcomes in the satisfying event A will have $2\log n$ consecutive heads starting from one of the i positions, where i is ranging from 1 to $n - 2\log n + 1$. Let us denote the range $[1, n + 1 - 2\log n]$ with R, for convenience. This therefore implies that,

$$A = \bigcup_{i \in [R]} A_i$$

Now, we can use **Union Bound** on A to calculate the required probability. Let $P(\cdot)$ denote the probability of an event happening. Using union bound we have,

$$P(A) \leq \sum_{i \in [R]} P(A_i)$$

Also, it is easy to see that $P(A_i) = 1/(2^{2\log n})\ \forall\ i\ \in\ [R]$, since it just says to place $2\log n$ consectutive heads starting from ith place and rest of the tosses can be anything. Plugging this above, we get

$$\begin{aligned} P(A) &\leq \sum_{i \in [R]} 1/(2^{2\log n}) \\ P(A) &\leq \frac{n + 1 - 2\log n}{2^{2\log n}} \\ P(A) &\leq \frac{n + 1 - 2\log n}{n^2} \\ P(A) &\leq \frac{1}{n} \end{aligned}$$

Since, we are asked to find the complement of event A in the original problem. Hence, we get

$$\begin{aligned} P(A^c) &= 1 - P(A) \\ P(A^c) &\geq 1 - 1/n \end{aligned}$$

Hence proved. $\square$

II Problem 2 Solution

In this problem, we are asked to find probability of some event conditioned on some other event. We will solve this using rules of conditional probability and also the fact that sum of an event and it's complement conditioned on some other event is 1. The two parts have been solved in the two points enumerated below.

1. Let $P(\cdot)$ denote the probability of an event, then

$$\begin{aligned}
P(\text{A to B open} \mid \text{No route from A to C}) &= 1 - P(\text{A to B closed} \mid \text{No route from A to C}) \\
&= 1 - \frac{P(\text{A to B closed} \cap \text{No route from A to C})}{P(\text{No route from A to C})} \\
&= 1 - \frac{p^2}{P(\text{A to B closed} \cup \text{B to C closed})} \\
&= 1 - \frac{p^2}{P(\text{A to B closed}) + P(\text{B to C closed})} \\
&\quad - P(\text{A to B closed} \cap \text{B to C closed}) \\
&= 1 - \frac{p^2}{p^2 + p^2 - (p^2)(p^2)} \\
&= 1 - \frac{1}{2 - p^2} \\
&= \frac{1 - p^2}{2 - p^2}
\end{aligned}$$

2. Let $P(\cdot)$ denote the probability of an event, then

$$\begin{aligned}
P(\text{A to B open} \mid \text{No route from A to C}) &= 1 - P(\text{A to B closed} \mid \text{No route from A to C}) \\
&= 1 - \frac{P(\text{A to B closed} \cap \text{No route from A to C})}{P(\text{No route from A to C})} \\
&= 1 - \frac{p^2 \cdot p}{P((\text{A to B closed} \cup \text{B to C closed}) \cap \text{A to C closed})} \\
&= 1 - \frac{p^2}{(P(\text{A to B closed}) + P(\text{A to C closed})} \\
&\quad - P(\text{A to B closed} \cap \text{A to C closed})) \cdot P(\text{A to C closed}) \\
&= 1 - \frac{p^3}{(p^2 + p^2 - (p^2)(p^2)) \cdot p} \\
&= 1 - \frac{1}{2 - p^2} \\
&= \frac{1 - p^2}{2 - p^2}
\end{aligned}$$

III Problem 3 Solution

In this problem we were asked to find the ratio of males and females in the next generation. We will calculate the expected number of males and females and then calculate the required ratio.

Let's first calculate the expected number of males. Since, next generation in all the families is independent, the total expected number of males and females will be n times the expected number in one family and hence while calculating the final ratio, this n will not matter. Hence, we will only work with expected values for one family.

Let X denote the number of males in next generation. Since, X can only take either value 0 or 1, it is easy to calculate it's expectation. We will calculate the probability of $X = 1$ by summing over the events that ith child is Male.

$$E[X] = P(\mathrm{X} = 1)$$

$$E[X] = \sum_{i=1}^{10} \frac{1}{2^i}$$

$$E[X] = 1 - \frac{1}{2^{10}}$$

Let Y denote the number of females in next generation. We will calculate the expectation by summing over the events with i number of females in next generation. In each term in this sum, the probability for $Y = i$ would be for probability of a sequence of i girls and then male occurring except when number of girls are 10.

$$E[Y] = \sum_{i=1}^{10} P(\mathrm{X} = \mathrm{i}) \cdot i$$

$$E[Y] = \sum_{i=1}^{9} \frac{i}{2^{i+1}} + \frac{10}{2^{10}}$$

$$E[Y] = \frac{1}{2} \cdot \left(\sum_{i=1}^{9} \frac{i}{2^i} \right) + \frac{10}{2^{10}}$$

$$E[Y] = \frac{1}{2} \cdot \left(\sum_{i=1}^{9} \frac{i}{2^{i-1}} - \frac{i}{2^i} \right) + \frac{10}{2^{10}}$$

On solving the telescopic sum in the end, we end up with

$$
\begin{aligned}
E[Y] &= \frac{1}{2}\cdot\left(\sum_{i=0}^{8}\frac{1}{2^i}\right) - \frac{9}{2^{10}} + \frac{10}{2^{10}} \\
E[Y] &= \frac{1}{2}\cdot\left(2 - \frac{1}{2^8}\right) + \frac{1}{2^{10}} \\
E[Y] &= 1 - \frac{1}{2^9} + \frac{1}{2^{10}} \\
E[Y] &= 1 - \frac{1}{2^{10}}
\end{aligned}
$$

Therefore, we get the expected ratio to be $E[X]:E[Y] = \frac{1-\frac{1}{2^{10}}}{1-\frac{1}{2^{10}}} = 1:1$.

IV Problem 4 solution

In this problem, we were asked to find the distribution of a random variable given in terms of two other random variables. We are given in the question that

$$f_X(x) = P(X \leq x) \tag{IV1}$$
$$f_Y(y) = P(Y \leq y) \tag{IV2}$$
$$Z = min(X, Y) \tag{IV3}$$

and we are asked to find out $f_Z(z) = P(Z \leq z)$ for z being some value in the range of Z. We know that sum of probabilities of random value Z taking some value from it's range will be 1 (similar is the case with random variable X and Y) and using this fact, we can easily solve this problem.

$$P(Z \leq z) + P(Z > z) = 1$$
$$P(Z \leq z) = 1 - P(Z > z)$$

Now using the fact that $Z = min(X, Y)$, we know that if Z is taking a value greater than some value z than both X and Y should take value greater than z. Therefore, we have

$$P(Z > z) = P(X > z \cap Y > z)$$
$$P(Z > z) = P(X > z) \cdot P(Y > z) \qquad \text{(X and Y are independent)}$$
$$P(Z > z) = (1 - f_X(z)) \cdot (1 - f_Y(z)) \qquad \text{(using (IV1) and (IV2))}$$

Therefore, we have

$$P(Z \leq z) = 1 - (1 - f_X(z)) \cdot (1 - f_Y(z))$$
$$P(Z \leq z) = f_X(z) + f_Y(z) - f_X(z) f_Y(z)$$
$$\therefore f_Z(z) = f_X(z) + f_Y(z) - f_X(z) f_Y(z)$$

Since, z can be any other variable, thereofer

$$f_Z = f_X + f_Y - f_X \cdot f_Y$$

4.1 Problem 5 solution

In this problem, we were asked to find out expected number of cycles in a random bijection Q. Let us define some events first

$$\begin{aligned} X &\rightarrow \text{Number of cycles in some bijection} Q, \\ X_i &\rightarrow \text{Number of cycles of length } i \text{ in } Q \end{aligned}$$

Trivially, the length of maximum cycle in Q can be n and minimum length can be 1. Therefore, we can see that

$$X = \sum_{i=1}^{n} X_i$$

Using linearity of expectation, we have that

$$E[X] = \sum_{i=1}^{n} E[X_i]$$

Let us calculate the $E[X_i]$ for any general i. For this i, let us generate set containing all the subsets of cardinality i. Let us name this set as S. There will be $\binom{n}{i}$ number of subsets in S. Let $S = \{S_1, S_2, S_3, \ldots\}$ where each S_j has cardinality i and there are $\binom{n}{i}$ of them in S. Let us define some events based on the above definition

$$\begin{aligned} A_j \rightarrow\ & 1 \text{ if elements from } S_j \text{ form a complete cycle} \\ & 0 \text{ otherwise} \end{aligned}$$

Therefore using the definitions set above, we have

$$X_i = \sum_{i \in [\binom{n}{i}]} A_i$$

Therefore using linearity of expectation, we have

$$E[X_i] = \sum_{j \in [\binom{n}{i}]} E[A_j]$$

Now, we will calculate the expected value of A_j using the standard formula of expectation

$$\begin{aligned} E[A_j] &= P(A_j = 1) \cdot 1 \\ &= \frac{(\text{Number of permutations inside } S_j) \cdot (n-i)!}{n!} \end{aligned}$$

Notice, that all the elements inside one complete cycle can be visualised as being points on a circle as this cycle can be represented as a directed cyclic graph. Hence, the problem of finding number of permutations inside S_j is same as the problem of finding number of permutations of points on a circle which is just $(i-1)!$ (Fixing one of the number and permuting the rest of the $j-1$ positions). Hence, we get

$$E[A_j] = \frac{(i-1)! \cdot (n-i)!}{n!}$$

Since, the expectation of A_j is independent of j, we have

$$E[X_i] = \sum_{j \in [\binom{n}{i}]} E[A_j]$$
$$E[X_i] = \sum_{j \in [\binom{n}{i}]} \frac{(i-1)! \cdot (n-i)!}{n!}$$
$$E[X_i] = \binom{n}{i} \cdot \frac{(i-1)! \cdot (n-i)!}{n!}$$
$$E[X_i] = \frac{n!}{(i)! \cdot (n-i)!} \cdot \frac{(i-1)! \cdot (n-i)!}{n!}$$
$$E[X_i] = \frac{1}{i}$$

Plugging this in the original equation, we get

$$E[X] = \sum_{i=1}^{n} E[X_i]$$
$$E[X] = \sum_{i=1}^{n} \frac{1}{i}$$

Hence, the expected number of cycles turn out to be the harmonic sum of numbers from 1 to n.