CS711: Introduction to Game Theory and Mechanism Design	**Jul-Nov 2020**

CS711: Mid Sem Home Assignment

1.1 Question 1

FALSE

Let the best utility of player i be B_i and let the dominant strategy for player i be s_i^*. It can be seen that there will exist some s_{-i} such that $u_i(s_i^*, s_{-i}) = B_i$ (B_i will definitely be corresponding to some strategy profile where player i plays s_i^* otherwise it wouldn't have been the dominant strategy for player i). But this doesn't necessarily mean that s_{-i} constitutes the **PSNE** for all the other players except player i.

A simple example is that of Neighbouring Kingdom's Dilemma (as given in Question 2). **Defense** is the dominant strategy for both player 1 and player 2. Therefore (**D,D**) is the **SDSE** and consequently the **PSNE**. The best utility in the game for player 1 is 6 but in the equilibrium he/she gets only 1 as the utility.

1.2 Question 2

The resulting game will have the following utilities' table-

	A	D
A	$(5+5\alpha, 5+5\alpha)$	$(6\alpha, 6)$
D	$(6, 6\alpha)$	$(1+\alpha, 1+\alpha)$

a. For $\alpha = 1$, the utilities of each action profile will be given by the following table-

	A	D
A	10,10	6,6
D	6,6	2,2

Clearly, A is the dominating strategy for both the players and hence, the reasonable outcome of the game is given by strategy profile (A,A). In contrast,in a classical Neighboring Kingdoms' Dilemma game, D is the dominating strategy for both the player and the reasonable outcome is given by the strategy profile (D,D).
Hence, this game is no longer a classical Neighboring Kingdoms' Dilemma.

b. For the resulting game to be the classical Neighboring Kingdoms' Dilemma, the strategy D must be dominating for both the players. This condition gives us the following inequalities-

$$\alpha + 1 \geq 6\alpha$$

$$6 \geq 5 + 5\alpha$$

By solving the inequalities, we have: for the resulting game to be the classical Neighbouring Kingdoms' Dilemma, $\alpha \leq 1/5$.
For values of α for which the game is not the Neighboring Kingdoms' Dilemma i.e. $\alpha > 1/5$,

- (D,D) cannot be the PSNE, as $u_2(D, A) > u_2(D, D)$ for $\alpha > 1/5$.
- (A,D) cannot be the PSNE, as for $\alpha > 1/5$, $u_1(A, A) = 5 + 5\alpha > 6 = u_1(D, A)$.
  Since the game is symmetric for both players, we can give an analous argument to prove that (A,D) is not a PSNE.
- (A,A) is a PSNE as for $\alpha > 1/5$, $u_1(A, A) = 5 + 5\alpha > 6 = u_1(D, A)$ and $u_2(A, A) = 5 + 5\alpha > 6 = u_2(A, D)$.

Hence, (A,A) is the only Nash equilibrium, for the values of α for which the game is not the Neighboring Kingdoms' Dilemma.

1.3 Question 3

The utility table is given by :

	Turn	Don't Turn
Turn	$(0, 0)$	$(-1, 1)$
Don't Turn	$(T, -1)$	$(-2, -2)$

Let us denote strategies Don't Turn by DTu and Turn by Tu

a. It is easy to see that (DTu, Tu) is a PSNE if $T > 0$. This is because we have,

$$u_1(DTu, Tu) = T > 0 = u_1(Tu, Tu) \tag{1.1}$$
$$u_2(DTu, Tu) = -1 > -2 = u_2(DTu, DTu) \tag{1.2}$$

Similarly, It can be shown that (Tu, DTu) is a PSNE too. The other two strategies aren't a PSNE because they violate some property of a PSNE.

b. When $T < 0$, we can see that (Tu, DTu) is a PSNE because we have,

$$u_1(Tu, DTu) = -1 > -2 = u_1(DTu, DTu) \tag{1.3}$$
$$u_2(Tu, DTu) = 1 > 0 = u_2(Tu, Tu) \tag{1.4}$$

The other three strategies aren't a PSNE because they violate some property of a PSNE.

c. Consider the mixed strategy $\sigma = (\sigma_1, \sigma_2)$ where $\sigma_1 = (p, 1 - p)$ and $\sigma_2 = (q, 1 - q)$. By the MSNE characterization theorem, we must have for player one,

$$u_1(DTu, \sigma_2) = u_1(Tu, \sigma_2) \qquad \textit{giving,} \tag{1.5}$$
$$(1 - q)(-1) = qT + (1 - q)(-2) \tag{1.6}$$
$$1 - q = qT \tag{1.7}$$
$$q = \frac{1}{T + 1} \tag{1.8}$$

and for player 2 we have,

$$\begin{aligned} u_2(DTu, \sigma_1) &= u_2(Tu, \sigma_1) \qquad giving, & (1.9)\\ (1-p)(-1) &= p + (1-p)(-2) & (1.10)\\ 1-p &= p & (1.11)\\ p &= \frac{1}{2} & (1.12) \end{aligned}$$

The utility in such an MSNE for player 1 is given by

$$(1-q)(-1) = \frac{-T}{1+T} \qquad (1.13)$$

and for player 2 is given by

$$(1-p)(-1) = \frac{-1}{2} \qquad (1.14)$$

d. with $T = 2$ we have $p = \frac{1}{2}$ and $q = \frac{1}{1+T}$ giving $q = \frac{1}{3}$. The payoffs for each of the players are $\frac{-2}{3}$ and $\frac{1}{2}$. Hence, player 1 has a more chance of turning. The higher expected payoff occurs for player 2. Player 2's Mixed strategy depends on T.

e. We see that if $T > 1$, Player 2 has a higher probability mass on DTu, but in turn gets a lower payoff than what player 1 gets $\frac{-1}{2}$. This is counter intuitive in the sense that player 2 doesn't focus on winning. This happens because the rules of the MSNE force Player 2 to put more weight on DTu in order to make have same expected payoff for both strategies of player 1

1.4 Question 4

		Player II		
		L	*C*	*R*
Player I	*T*	3	−3	0
	M	2	6	4
	B	2	5	6

a. Assume that the mixed strategy of player 1 $\sigma_1 = \{p, q, 1-p-q\}$ be such that it guarantees him the same payoff against any pure strategy of Player II i.e.

$$u_1(\sigma_1, L) = u_1(\sigma_1, C) = u_1(\sigma_1, R)$$

$$\implies 3p + 2q + 2(1-p-q) = -3p + 6q + 5(1-p-q) = 0 + 4q + 6(1-p-q)$$

$$\implies p = 2/5, q = 3/5, 1-p-q = 0$$

Hence the required mixed strategy for player 1 is {2/5,3/5,0}.

b. Assume that the mixed strategy of player 2 $\sigma_2 = \{p, q, 1-p-q\}$ be such that it guarantees him the same payoff against any pure strategy of Player I i.e.

$$u_2(T, \sigma_2) = u_2(M, \sigma_2) = u_2(B, \sigma_2)$$

But since $u_2(s) = -u_1(s) \forall s \epsilon S$, so we have

$$u_1(T, \sigma_2) = u_1(M, \sigma_2) = u_1(B, \sigma_2)$$

$$\implies 3p - 3q + 0(1-p-q) = 2p + 6q + 4(1-p-q) = 2 + 5q + 6(1-p-q)$$

$$\implies p = 22/25, q = 2/25, 1-p-q = 1/25$$

Hence the required mixed strategy for player 2 is {22/25,2/25,1/25}.

c. The MSNE for this game is ((2/5,5/3,0),(22/5,2/25,1/25)). We have already proved in Lectures, that the notion of Nash equilibrium and the notion of minmax equilibrium is the same. Hence, the strategies corresponding to the MSNE will also be the respective maxmin and minmax strategies. According to Minmax Theorem (we skip the proof of minmax theorem as it is essentially a repetition of Theorem 5.11 in the MSZ book), every game will have a value. Hence, by definition, the optimal strategies of the players will be the maxmin strategies and the minmax strategies. Since the minmax and maxmin strategies are obtained are the same as the strategies obtained in (4a) and (4b), therefore, the strategies obtained in 4(a) and 4(b) are also the optimal strategies.

d. We have already seen in lectures that the mixed strategy corresponding to MSNE strategies are calculated by equating the utilities of a player with different pure strategies of the opponent. Hence, trivially, if each player has an equating strategy, say σ_1, σ_2, then the strategy profile (σ_1, σ_2) is a MSNE. Since it is an MSNE, σ_1 and σ_2 must also be the maxmin and minmax mixed strategies of the game. As stated above, according to Minmax Theorem , every game will have a value. Hence, by definition, the optimal strategies of the players will be the maxmin strategies and the minmax strategies. Therefore, the optimal strategies will also be (σ_1, σ_2).

e. The given statement is not a contradiction to part d) because part (d) says that an equalising strategy exists for both the players. However, if only one of the players has an equalising strategy, the strategy need not necessarily be optimal. The example which further illustrates the point is given below.

	A	B	C	D
P	3	-3	0	1
Q	2	6	4	1
R	2	5	6	1

1.5 Question 5

The following is the game matrix with the payoffs of Army A (as it is a zero sum game, the payoffs of Army B is the negative of the payoffs of Army A). The actions of Army A are along the rows and for Army B are along the columns.

	1	2	3
1	0	v_1	v_1
2	v_2	0	v_2
3	v_3	v_3	0

Here, $v_1 > v_2 > v_3$
Now, let (p^*, q^*) represent a MSNE, where $p^* = (p_1^*, p_2^*, p_3^*), p_1^*+p_2^*+p_3^* = 1$ and $q^* = (q_1^*, q_2^*, q_3^*), q_1^*+q_2^*+q_3^* = 1$.

We see that if Army A does not attack a target i, that is a strategy i is not in the support of Army 1 (that is $p_i^* = 0$), then Army B does not need to defend the target i, and hence strategy i is not in the support of Army B also, and thus $q_i^* = 0$. Now, if $q_i^* = 0$, and $i < 3$, then $p_{i+1}^* = 0$, as Army A will achieve a higher payoff by attacking target i instead of $i + 1$.
Thus, if Army A does not attack target i, then it will surely not attack target $i + 1$

Thus, the possibilities are that Army A attacks:

- **Only target 1:** If Army A attacks only target 1, then Army B can defend target 1, and the utility for Army A will clearly be more if it attacks target 2 or 3, given that Army B defends target 1 (which dissatisfies the condition for a mixed strategy profile to be a MSNE). Thus, there is **no MSNE in this case.**

- **Target 1 and 2, but not 3:** Both A and B should be indifferent to use strategies 1 or 2 with non negative probabilities, and should not play strategy 3. Thus let $p^* = (p, 1-p, 0)$ and $q^* = (q, 1-q, 0)$. Then, we have

$$(1-p)v_2 = pv_1 \tag{1.15}$$
$$\therefore p = \frac{v_2}{v_1+v_2}, \tag{1.16}$$
$$(1-q)v_1 = qv_2 \tag{1.17}$$
$$\therefore q = \frac{v_1}{v_1+v_2} \tag{1.18}$$

  Clearly, p, q are non negative.
  Thus $p^* = \left(\frac{v_2}{v_1+v_2}, \frac{v_1}{v_1+v_2}, 0\right),\ q^* = \left(\frac{v_1}{v_1+v_2}, \frac{v_2}{v_1+v_2}, 0\right)$

  As A does not attack target 3, the utility in attacking target 3 should not be greater than the expected utility for A in this case, and thus $v_3 \leq \frac{v_1 v_2}{v_1+v_2}$ for (p^*, q^*) to be an MSNE in this case.

- **Target 1, 2 and 3:** Both A and B should be indifferent to use strategies 1 or 2 or 3 with non negative probabilities. Then, let $p^* = (p_1, p_2, p_3), q^* = (q_1, q_2, q_3)$, we have

$$p_2v_2 + p_3v_3 = p_1v_1 + p_3v_3 = p_1v_1 = p_2v_2 \tag{1.19}$$
$$\implies p_1v_1 = p_2v_2 = p_3v_3 \tag{1.20}$$
$$\because p_1 + p_2 + p_3 = 1 \tag{1.21}$$
$$\therefore p_1 = \frac{v_2v_3}{S}, p_2 = \frac{v_1v_3}{S}, p_3 = \frac{v_1v_2}{S}, \tag{1.22}$$
$$q_2v_1 + q_3v_1 = q_1v_2 + q_3v_2 = q_1v_3 + q_2v_3 \tag{1.23}$$
$$\implies (1-q_1)v_1 = (1-q_2)v_2 = (1-q_3)v_3 \tag{1.24}$$
$$\because q_1 + q_2 + q_3 = 1 \implies 1 - q_1 + 1 - q_2 + 1 - q_3 = 2 \tag{1.25}$$
$$\therefore q_1 = \frac{S - 2v_2v_3}{S}, q_2 = \frac{S - 2v_1v_3}{S}, q_3 = \frac{S - 2v_1v_2}{S} \tag{1.26}$$

  where $S = v_1v_2 + v_2v_3 + v_1v_3$.

Thus, $p^* = \left(\frac{v_2v_3}{S}, \frac{v_1v_3}{S}, \frac{v_1v_2}{S}\right)$ and $q^* = \left(\frac{S-2v_2v_3}{S}, \frac{S-2v_1v_3}{S}, \frac{S-2v_1v_2}{S}\right)$

Thus, p_1, p_2, p_3 are clearly ≥ 0 and we also need to ensure that $q_1, q_2, q_3 \geq 0$, and as q_3 is the least of these, $q_2 \geq 0 \implies S - 2v_1v_2 \geq 0 \implies v_3 \geq \frac{v_1v_2}{v_1+v_2}$, for (p^*, q^*) to be an MSNE in this case.

1.6 Question 6

1.6.1 Part (a):

We can formulate this situation as a normal form game as follows:

- The set of players consists of two players, say 1 and 2. Therefore $N = 1, 2$.
- The set of strategy set of each player consists of choosing the amount of effort for each player. Therefore, the effort of each player is equivalent to the action taken by the player.
- The utility function of each player can be defined in the following way.

$$u_i(x_1, x_2) = \frac{f(x_1, x_2)}{2} - c(x_i) \tag{1.27}$$

The above three points allow us to formulate the given situation as NFG.

1.6.2 Part (b):

1. $f(x_1, x_2) = 3x_1x_2$ and $c(x_i) = x_i^2$, i = 1,2
   Aim: For player 1, maximise $u_1(x_1, x_2)$ with respect to x_1.

$$\max_{x_1} \; u_1(x_1, x_2) \tag{1.28}$$

$$\implies \max_{x_1} \; \frac{3x_1x_2}{2} - x_1^2 \tag{1.29}$$

   The first order conditions of optimality for player 1 is

$$\frac{\partial u_1(x_1, x_2)}{\partial x_1} = 0 \tag{1.30}$$

$$\implies \frac{3x_2}{2} - 2x_1 = 0 \tag{1.31}$$

   A similar analysis for player two yields it's condition of optimality as

$$\frac{3x_1}{2} - 2x_2 = 0 \tag{1.32}$$

   Solving equations (1.19) and (1.18) simultaneously yields the solution

$$x_1 = 0, x_2 = 0 \tag{1.33}$$

   Therefore the above action set is the nash equilibria for the following game.

2. $f(x_1, x_2) = 4x_1x_2$ and $c(x_i) = x_i$, $i = 1{,}2$
   Aim: For player 1, maximise $u_1(x_1, x_2)$ with respect to x_1.

$$\max_{x_1} \; u_1(x_1, x_2) \tag{1.34}$$

$$\implies \max_{x_1} \; \frac{4x_1x_2}{2} - x_1^2 \tag{1.35}$$

   The first order conditions of optimality for player 1 is

$$\frac{\partial u_1(x_1, x_2)}{\partial x_1} = 0 \tag{1.36}$$

$$\implies 2x_2 - 1 = 0 \tag{1.37}$$

   A similar analysis for player two yields it's condition of optimality as

$$2x_1 - 1 = 0 \tag{1.38}$$

   Solving equations (1.19) and (1.18) simultaneously yields the solution

$$x_1 = \frac{1}{2}, x_2 = \frac{1}{2} \tag{1.39}$$

   Therefore the above action set is the nash equilibria for the following game.

1.6.3 Part (c):

If we plug in the nash equilibrium strategy values in each of the above two cases respectively, we get that utilities for each player in both the cases are equal to 0.

Clearly, choosing an effort set such as $x_1 = 1$ and $x_2 = 1$ pays a positive utility to each of the player in both of the cases. Therefore, we can say that **YES** there exists a pair of efforts which yields higher payoffs than the Nash equilibrium effort.

1.7 Question 7

Given : A two-player, symmetric game which has a PSNE (s_1, s_2).
To Prove : (s_2, s_1) is also a PSNE.
Proof : (s_1, s_2) being a PSNE implies that

$$u_1(s_1, s_2) \geq u_1(s_1^{'}, s_2) \qquad \forall s_1^{'} \in S_1 \tag{1.40}$$

$$u_2(s_1, s_2) \geq u_2(s_1, s_2^{'}) \qquad \forall s_2^{'} \in S_1 \tag{1.41}$$

Using the property of symmetry, we get from the first and second equations above,

$$u_2(s_2, s_1) \geq u_2(s_2, s_1^{'}) \qquad \forall s_1^{'} \in S_1 \tag{1.42}$$

$$u_1(s_2, s_1) \geq u_1(s_2^{'}, s_1) \qquad \forall s_2^{'} \in S_1 \tag{1.43}$$

The above two equations, imply that (s_2, s_1) is a PSNE too.

1.8 Question 8

Given: A normal form game $< N, (A_i)_{i \in \mathbb{N}}, (u_i)_{i \in \mathbb{N}} >$ such that $\exists \phi \;:\; A \;\rightarrow \mathbb{R}$ such that for every player $i \in N$, for all $a_i, a_i' \in A_i$ and for all $a_{-i} \in A_{-i}$

$$u_i(a_i, a_{-i}) - u_i(a_i', a_{-i}) = \phi(a_i, a_{-i}) - \phi(a_i', a_{-i}) \tag{1.44}$$

To Prove: This game has a pure strategy Nash equilibrium.

Proof: Observe a few special points about the function ϕ.

- ϕ assigns a real value for every $s \in S$.
- By assuming any arbitrary value as reference, we can construct a matrix for this function ϕ as well where each entry corresponds to the vector of strategies from the original utility matrix.

  For example :

	A	D
A	1,1	10,0
D	0,5	7,2

	A	D
A	4	3
D	3	0

  Here the first matrix represents the game matrix and second matrix represents the corresponding ϕ matrix.

  Note: This represents just one out of many possible ϕ matrices.

Claim: The global maximum of ϕ matrix is a **PSNE** of the original game.

Proof: To prove this, suppose s^* corresponds to global maximum of ϕ matrix. Then, for any $i \in N$, we have by definition of global maximum that

$$\phi(s^*, s_{-i}^*) - \phi(s, s_{-i}^*) \geq 0 \tag{1.45}$$

$$\implies u_i(s^*, s_{-i}^*) - u_i(s, s_{-i}^*) \geq 0 \; \forall \; s_i \in S_i \; and \; \forall i \in N \tag{1.46}$$

Hence, s^* is the PSNE of the original game. Hence, the claim is true. ■

By using the above claim, we proved that there exists atleast one **PSNE** in the original game. ■

1.9 Question 9

As \$0.20 are removed after every round, we can clearly see that the game has to end after 5 rounds, as no money (that is \$ 0.00) will be left to split up. So, it is clear that the leaf nodes are reached in this round. So we find the SPNE by performing backward induction from this round. As it is an odd numbered round, Player 1 makes the offer.
At any round, we represent the offered split of money as (\$a, \$b), where \$a is the money received by Player 1 and \$b is the money received by Player 2.

- **Round 5:** Player 1 makes an offer of \$0.00 to Player 2. Now, if Player 2 rejects the offer, then he would still get \$0.00 in the next round (as there will be no more rounds), so there is not benefit in rejecting the offer. So he accepts it. Player 1 thus keeps \$0.20 to himself. The offered split is hence (\$0.20, \$0.00).

- **Round 4:** In this round, Player 2 makes an offer. He knows that in the next round, the best amount Player 1 can get is \$0.20. Thus he makes an offer with the split (\$0.20, \$0.20), and Player 1 has to accept because he can't get better than \$0.20 in the next round.
- **Round 3:** Similar to previous rounds, Player 1 offers the split (\$0.40, \$0.20), which is accepted by Player 2, as he can't get more than \$0.20 in the next round, by rejecting this offer.
- **Round 2:** Player 2 offers the split (\$0.40, \$0.40), which is accepted by Player 1 (reason same as above arguments)
- **Round 1:** Player 1 offers the split (\$0.60, \$0.40), accepted by Player 2.

These result of backward induction can be represented by the SPNE (p^*, q^*), where p^* is the strategy of Player 1 and q^* is the strategy of player 2, such that:

p^* = (Offer (\$0.60, \$0.40), Accept if offered $\geq$ \$0.40, Offer (\$0.20, \$0.20), Accept if offered $\geq$ \$0.20, Offer (\$0.20, \$0.00))

q^* = (Accept if offered $\geq$ \$0.40, Offer (\$0.40, \$0.40), Accept if offered $\geq$ \$0.20, Offer (\$0.20, \$0.20), Accept if offered $\geq$ \$0.00)

1.10 Question 10

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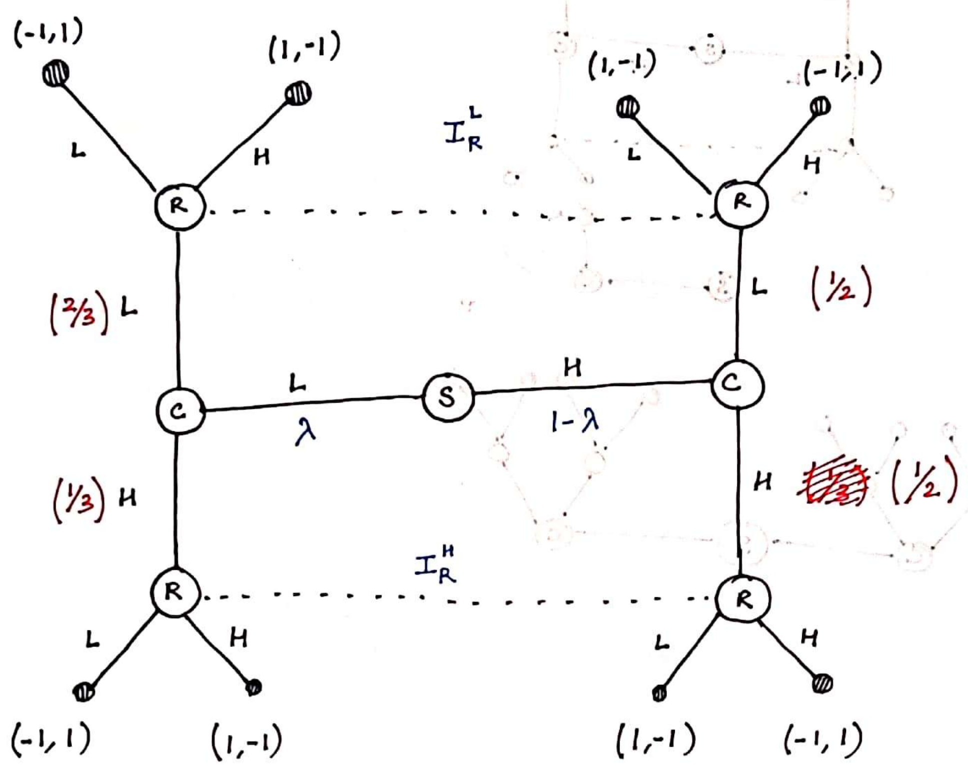
Tuple for utility is of the form (S, R) ← utility of S : utility of R



* Let player randomise in the set I_R^H as $(L, H) = (x, 1-x)$.
Similarly, in I_R^L, player R randomises over (L, H) as $(y, 1-y)$

* Let player "S" mix over his information set as follows:
$(L, H) = (\lambda, 1-\lambda)$

→ Intuition: Suppose we get rid of C, we can notice that a rough NFG representation of this looks like

S \ R	L	H
L	-1, 1	1, -1
H	1, -1	-1, 1

We know that in such a zero sum game, there does not exist any PSNE.
The best we ~~can~~ can get is an MSNE where the expected utility of both the players is 0.

So, in this PBE also, we will construct something similar.

→ Notice that if the expected utility of player S is V, then the expected utility of player R is −V.

* Expected utility of player R, given C = L :

$$U_{R, C=L} = \lambda \times \frac{2}{3} \times [y(1) + (1-y)(-1)] + \frac{1-\lambda}{2} \times [y(-1) + (1-y)]$$

$$= (2y-1)\left[\frac{2\lambda}{3} + \frac{\lambda}{2} - \frac{1}{2}\right] = (2y-1)\left[\frac{7\lambda - 3}{6}\right] \quad (1)$$

* Similarly, expected utility of player R, given C = H,

$$U_{R,C=H} = \lambda \times \frac{1}{3} \times [x(1) + (1-x)(-1)] + \frac{1-\lambda}{2} \times [x(-1) + (1-x)]$$

$$= (2x-1)\left[\frac{\lambda}{3} + \frac{\lambda}{2} - \frac{1}{2}\right] = (2x-1)\left[\frac{5\lambda - 3}{6}\right] \quad (2)$$

By differentiating wrt x, y & λ, we will get some conditions

$$\frac{\partial U_{R,C=L}}{\partial y} = 0 \Rightarrow \lambda = \frac{3}{7} \quad ; \quad \frac{\partial U_{R,C=L}}{\partial \lambda} = 0 \Rightarrow y = \frac{1}{2}$$

$$\frac{\partial U_{R,C=H}}{\partial x} = 0 \Rightarrow \lambda = \frac{3}{5} \quad ; \quad \frac{\partial U_{R,C=H}}{\partial \lambda} = 0 \Rightarrow x = \frac{1}{2}$$

→ We can only make do with a best utility of 0

∴ Case I: $\lambda = \frac{3}{5}$, $y = \frac{1}{2}$ & x can vary

Case II: $\lambda = \frac{3}{7}$, $x = \frac{1}{2}$ & y can vary.