CS711: Introduction to Game Theory and Mechanism Design

Midsem take-home exam – Semester 1, 2020-21
Computer Science and Engineering
Indian Institute of Technology Kanpur
Total Points: 100, Time: 8 hours, ATTEMPT ALL QUESTIONS
Please submit your solutions as PDF files generated through LaTeX. See the course webpage for LaTeX tutorials and submission template. Begin the solution of a question in a new page. This is a group assignment. Please submit only one solution PDF named {group number}.pdf, e.g., 13.pdf for group 13, from each group (also write the group number and the names of the members in the submission file). Please email it to `swaprava@cse.iitk.ac.in` with a cc to `garima@cse.iitk.ac.in` with the subject "[CS711] Midsem take-home exam submission". There is no need to write the question again in the solution.

1. Is the following statement `true` or `false`? "If a player has a dominant strategy in a simultaneous-move game, then she is sure to get her best possible utility in any Nash equilibrium of the game." Explain your answer and provide an example of a game that illustrates your answer. **2+8 points.**

2. In the classical Neighboring Kingdoms' Dilemma game, each of two players had two possible actions, agriculture (A) and defense (D). The utilities for each action profile are given in the following table:

		Player 2	
		A	D
Player 1	A	5, 5	0, 6
	D	6, 0	1, 1

Consider an altruistic variation of the game where each player not only cares about their own payoff, but also the other player's payoff. In particular, each player's modified payoff becomes his original payoff plus α times the original payoff of the other player. For example, player 1's modified payoff to action profile (A, A) is $5 + 5\alpha$ and payoff to action profile (A, D) is $0 + 6\alpha$.

(a) Write down the strategic form of this game for $\alpha = 1$. Is this game still a classical Neighboring Kingdoms' Dilemma game (in terms of the conclusions about the outcome)? Explain your answer. **1+2 points.**

(b) Find the range of values of α for which the resulting game is the classical Neighboring Kingdoms' Dilemma. For values of α for which the game is not the Neighboring Kingdoms' Dilemma, find all its Nash equilibria. **2+5 points.**

3. Consider the game represented in the table below, where Player 1 chooses the row and Player 2 chooses the column.

	Turn	Don't Turn
Turn	0,0	-1,1
Don't Turn	T,-1	-2,-2

   (a) Find all of the pure strategy Nash equilibria for this game if $T > 0$.

   (b) Find all of the pure strategy Nash equilibria for this game if $T < 0$.

   (c) If $T > 0$, there is a mixed strategy Nash equilibrium strategy profile that is not a pure strategy Nash equilibrium. Find it and find the payoffs to each player in this equilibrium.

   (d) In a mixed strategy Nash equilibrium with $T = 2$, which player is more likely to turn? If $T = 2$, which player gets the higher expected payoff in equilibrium? Which player's equilibrium mixed strategy depends on T?

   (e) Is there anything paradoxical about the results in Parts (3c) and (3d)? If so, what?

2+1+2+3+2 points.

4. Consider the two-player *zero-sum* game in the figure below, in which each player has three pure strategies.

Player II

Player I		*L*	*C*	*R*
	T	3	−3	0
	M	2	6	4
	B	2	5	6

   (a) Find a mixed strategy of Player I that guarantees him the same payoff against any pure strategy of Player II.

   (b) Find a mixed strategy of Player II that guarantees him the same payoff against any pure strategy of Player I.

   (c) Prove that the two strategies you found in Parts (4a) and (4b) are the optimal strategies of the two players.

   (d) Generalize this result: Suppose a two-player zero-sum game is represented by an $n \times m$ matrix. Suppose each player has an *equalizing strategy*, meaning a strategy guaranteeing him the same payoff against any pure strategy his opponent may play. Prove that any equalizing strategy is an optimal strategy.

   (e) Give an example of a two-player zero-sum game in which one of the players has an equalizing strategy that is not optimal. Why is this not a contradiction to Part (4d)?

1+1+2+3+3 points.

5. **[Air strike].** Army A has a single plane with which it can strike one of three possible targets. Army B has one anti-aircraft gun that can be assigned to defend one of these targets. The value of target k is v_k, with $v_1 > v_2 > v_3 > 0$. Army A can destroy a target only if the target is undefended and A attacks it. Army A wishes to *maximize* the expected value of the damage (which is its payoff) and army B wishes to *minimize* it (hence it is their negative payoff). Formulate the situation as a (strictly competitive) normal form game and find its mixed strategy Nash equilibria. **5+5 points.**

6. Two people are engaged in a joint project. If each person i puts in the effort $x_i \in [0, 1]$, which costs her $c(x_i)$, the outcome of the project is worth $f(x_1, x_2)$. The worth of the project is split equally between the two people, regardless of their effort levels.

   (a) Formulate this situation as a normal-form game.

   (b) Find its Nash equilibria when

      i. $f(x_1, x_2) = 3x_1x_2,\ c(x_i) = x_i^2,\ i = 1, 2.$

      ii. $f(x_1, x_2) = 4x_1x_2,\ c(x_i) = x_i,\ i = 1, 2.$

   (c) In each case, is there a pair of effort levels that yields both players higher payoffs than the Nash equilibrium effort?

2+(3+3)+2 points.

7. A two-player game is symmetric if the two players have the same strategy set $S_1 = S_2$ and the payoff functions satisfy $u_1(s_1, s_2) = u_2(s_2, s_1)$ for each $s_1, s_2 \in S_1$. Prove that the set of PSNEs of a two-player symmetric game is a symmetric set: if (s_1, s_2) is a PSNE, then (s_2, s_1) is also a PSNE. **10 points.**

8. Consider a normal form game $\langle N, (A_i)_{i\in N}, (u_i)_{i\in N}\rangle$ where $\exists \phi : A \mapsto \mathbb{R}$ such that for every player $i \in N$, for all $a_i, a_i' \in A_i$ and for all $a_{-i} \in A_{-i}$

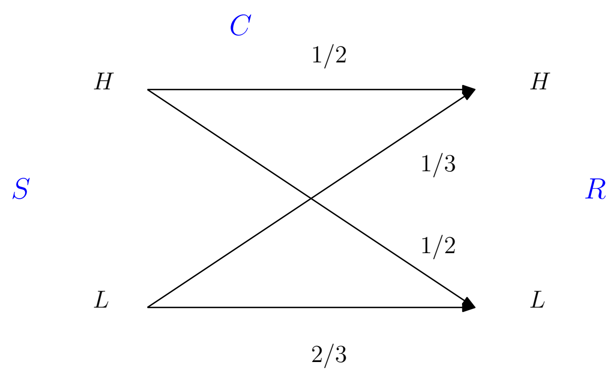
$$u_i(a_i, a_{-i}) - u_i(a_i', a_{-i}) = \phi(a_i, a_{-i}) - \phi(a_i', a_{-i}).$$

   Prove that this game has a pure strategy Nash equilibrium. You can assume that the strategy sets are finite for all players. **10 points.**

9. **[PIEFG].** Suppose that two players are bargaining over \$1. The game takes place in rounds, beginning with Round 1. The game ends when an offer is accepted. An offer is a proposed division of the remaining amount of money on table. Player 1 makes offers in odd-numbered rounds and Player 2 makes offers in even-numbered rounds. In every round, one player makes an offer and the other player and decides whether to "Accept" or "Reject". If accepted, the game ends, but if rejected, the player rejecting the offer makes a fresh offer on the remaining amount of money in the next round. At the end of each round, \$0.20 is removed from the pool

of money. That is, if an agreement is reached in Round 2, the total pool of money is \$0.80; if agreement in Round 3, \$0.60, and so forth. Find the subgame perfect Nash equilibrium of this bargaining game. **10 points.**

10. **[IIEFG].** Consider the following scenario with three players, a sender `S`, a channel `C`, and a receiver `R`. Player `S` has two actions: pick a high `H` or a low `L` signal, which is visible to `C` but not to `R`. Player `C` then transmits the signals to the receiver with probabilities as shown in the figure below (e.g., signal `L` is received as `H` with probability 1/3). The strategies of `C` is fixed and is a common knowledge.



Finally, player `R` guesses what signal was originally transmitted. If it guesses it correctly, the payoff to `R` is 1 and that to `S` is -1. If the guesses incorrectly, then the payoffs flip, i.e., -1 to `R` and 1 to `S`. Player `C` gets no payoff in any of these strategy profiles.

- Formulate this as an IIEFG and draw the game tree.
- Find the perfect Bayesian equilibria of this game.

5+5 points.

Good Luck!